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Interval-valued multi-fuzzy soft set and its application in decision making

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***Abstract*—**Multi-fuzzy set and soft set are effective mathematical tool for dealing with fuzzy and uncertainty, and they have been successfully applied to many field. In this paper, we first extend multi-fuzzy set to interval-valued multi-fuzzy set and study its some basic properties. Based on the soft set and interval-valued multi-fuzzy set, we propose a new hybrid model named interval-valued multi-fuzzy soft set, which can be regarded as a extension of some existing models. Some basic operations of interval-valued multi-fuzzy soft set are introduced and some of their properties are also discussed. Finally, we propose an approach to multiple attribute decision making problems based on interval-valued multi-fuzzy soft set and give an illustrative example to show the practicality and effectiveness of the proposed approach.

***Keywords*—**Soft set, Fuzzy soft set, Multi-fuzzy set, Interval-valued multi-fuzzy soft set, Decision making.

I. Introduction

Soft set and multi-fuzzy set first were introduced by Molodsov [1] and Sebastian [2], respectively, which are very effective to deal with uncertainties. Since their appearance, soft set and multi-fuzzy set have attracted the attention from many scholars at home and abroad. Lots of results on soft set and multi-fuzzy set have been acquired in theory and application. For example, Sezgin and Atagun [3] introduced some operations on soft set theory. Ali et al. [4, 5] further defined some new operations on soft set and studied their algebraic structures. Many scholars applied soft set to some algebraic structures and obtained new algebraic models, such as soft rings [6], soft groups [7] and soft BCK/BCI-algebras [8]. Soft set has also been applied to decision making and other fields [9–11].

In the study of soft set, it is a focus to how combine soft sets and other uncertain models. In this respect, many results have been acquired. By combining fuzzy set and soft set, Maji et al. [12] gave the concept of fuzzy soft set. Yang et al. [13] introduced interval-valued fuzzy soft set by combining interval-valued fuzzy set and soft set. Jiang et al. [14] further extended fuzzy soft set to intuitionistic fuzzy soft set. Yang et al. [15] introduced a new hybrid model named multi-fuzzy soft set combining soft set and multi-fuzzy set. Many other hybrid soft set models could be found in [16–19]. Motivated by this viewpoint, the main goal of this paper is to combine interval-valued fuzzy set, multi-fuzzy set and soft set and obtain a new hybrid model called interval-valued multifuzzy soft set, which can be viewed as an interval-valued extension of the multi-fuzzy soft set or a generalization of the interval-valued fuzzy soft set. We also introduce a adjustable approach to multiple attribute decision making based on interval-valued multi-fuzzy soft set.

The rest of this paper is organized as follows. Some basic concepts are briefly reviewed in Section II. The interval-valued multi-fuzzy set is introduced and some operations of interval-valued multi-fuzzy set are defined in Section III. In Section IV, we first give the concept of interval-valued multi-fuzzy soft set based on interval-valued multi-fuzzy set and soft set. we also define some operations of interval-valued multi-fuzzy soft set and discuss some basic properties. Section V develops an adjustable approach to interval-valued multi-fuzzy soft set based decision making and gives an example to illustrate the developed approach. Finally, Section VI presents the conclusion.

II. Preliminaries

In this section, we shall briefly review some basic concepts. In this work, let *U* be an initial universe of objects and *E* the set of parameters in relation to objects in *U*.

Definition 1. [1] A pair (F, A) is called a soft set over *U*, where $A \subseteq E$ and F is a set valued mapping given by $F : A \to P(U)$.

Definition 2. [12] Let $\mathrm{P}(U)$ be the set of all fuzzy subsets of *U*. A pair (F, A) is called a fuzzy soft set over *U*, where F is a set valued mapping given by $F : A \to \mathrm{P}(U)$.

Definition 3. [2] Let n be a positive integer. A multi-fuzzy set M in U is a set of ordered sequences having the form

$$M = \{u / (\mu_1(u), \mu_2(u), \cdots, \mu_i(u), \cdots, \mu_n(u)) : u \in U\},$$

where $\mu_i \in \mathrm{P}(U), i = 1, 2, \cdots, n.$

The function $\mu_M = (\mu_1, \mu_2, \cdots, \mu_n)$ is called the multi-membership function of multi-fuzzy set M, n is called the dimension of M. The set of all multi-fuzzy sets of dimension n in U is denoted by $\mathrm{M}^n(U)$.

Definition 4. [15] A pair (F, A) is called a multi-fuzzy soft set of dimension n over U, where F is a mapping given by $F: A \to \mathrm{M}^n(U)$.

Definition 5. [20] An interval-valued fuzzy set X on a universe U is a mapping $X: U \to Int([0,1])$, where $Int([0,1])$ stands for the set of all closed subintervals of $[0,1]$.

An interval-valued fuzzy set X can expressed as

$$X = \{u / \mu_X : u \in U\} = \{u / [\mu_X^-, \mu_X^+] : u \in U\},$$

where $0 \le \mu_X^-(u) \le \mu_X^+(u) \le 1$, $\mu_X^-(u)$ and $\mu_X^+(u)$ are referred to as the lower and upper degrees of membership of U to X, respectively. The set of all interval-valued fuzzy sets on U is denoted by $IVF(U)$.

Definition 5. [20] Let $X, Y \in IVF(U)$. Then some basic operations and relations on X, Y are given as follows:

(1) $X \cup Y = \{u / \mu_{X \cup Y} : u \in U\}$

$$= \{u / [max(\mu_X^-, \mu_Y^-), max(\mu_X^+, \mu_Y^+)] : u \in U\};$$

(2) $X \cap Y = \{u / \mu_{X \cap Y} : u \in U\}$

$$= \{u / [min(\mu_X^-, \mu_Y^-), min(\mu_X^+, \mu_Y^+)] : u \in U\};$$

(3) $X \subseteq Y \Leftrightarrow \mu_X^- \le \mu_Y^-$ and $\mu_X^+ \le \mu_Y^+$;

(4) $X^c = \{u / \mu_{X^c} : u \in U\}$

$$= \{u / [1 - \mu_X^+, 1 - \mu_X^-] : u \in U\}.$$

III. INTERVAL-VALUED MULTI-FUZZY SET

In this section, we extend multi-fuzzy set to interval-valued multi-fuzzy set for deal with the interval data, some basic operations and their properties are also studied.

Definition 7. Let n be a positive integer. A interval-valued multi-fuzzy set X in U is a set of ordered sequences

$$X = \{u / (\mu_1(u), \mu_2(u), \cdots, \mu_i(u), \cdots, \mu_n(u)) : u \in U\}$$
$$= \{u / ([\mu_1^-(u), \mu_1^+(u)], [\mu_2^-(u), \mu_2^+(u)], \cdots,$$
$$[\mu_i^-(u), \mu_i^+(u)], \cdots, [\mu_n^-(u), \mu_n^+(u)]) : u \in U\}$$

where $\mu_i \in IVF(U), i = 1, 2, \cdots, n.$

The function $\mu_X = (\mu_1, \mu_2, \cdots, \mu_n)$ is called the interval-valued multi-membership function of interval-valued multi-fuzzy set X, n is called the dimension of X. The set of all interval-valued multi-fuzzy sets of dimension n in U is denoted by $\mathrm{I\,VM}^n(U)$.

Remark 1. Clearly, an interval-valued multi-fuzzy set of dimension 1 is a interval-valued fuzzy set, and an interval-valued multi-fuzzy set of dimension 2 with $\mu_1^+(u) + \mu_2^+(u) \le 1$ is an interval-valued intuitionistic fuzzy set.

Remark 2. If $\sum_{i=1}^{n} \mu_i^+(u) \le 1$ for all $u \in U$, then the interval-valued multi-fuzzy set X of dimension n is called a normalized interval-valued multi-fuzzy set. Otherwise, X is non-normalized. If $\sum_{i=1}^{n} \mu_i^+(u) = l > 1$ for some $u \in U$, we divide the membership function μ_i by l, i.e. μ_i / l = $[\mu_X^- / l, \mu_X^+ / l]$, then the nonnormalized interval-valued multi-fuzzy set can be changed into a normalized interval-valued multi-fuzzy set.

Definition 8. Let $X = \{u / ([\mu_1^-(u), \mu_1^+(u)], [\mu_2^-(u), \mu_2^+(u)], \cdots, [\mu_n^-(u), \mu_n^+(u)]) : u \in U\}$ and $Y = \{u / ([\nu_1^-(u), \nu_1^+(u)], [\nu_2^-(u), \nu_2^+(u)], \cdots, [\nu_n^-(u), \nu_n^+(u)]) : u \in U\}$ be two interval-valued multi-fuzzy sets of dimension n in *U*.

(1) $X \cup Y = \{u / ([\mu_1^-(u) \vee \nu_1^-(u), \mu_1^+(u) \vee \nu_1^+(u)], \cdots,$
$[\mu_n^-(u) \vee \nu_n^-(u), \mu_n^+(u) \vee \nu_n^+(u)]) : u \in U\}$

(2) $X \cap Y = \{u / ([\mu_1^-(u) \wedge \nu_1^-(u), \mu_1^+(u) \wedge \nu_1^+(u)], \cdots,$
$[\mu_n^-(u) \wedge \nu_n^-(u), \mu_n^+(u) \wedge \nu_n^+(u)]) : u \in U\}$

(3) $X^c = \{u / ([1 - \mu_1^+(u), 1 - \mu_1^-(u)], \cdots,$
$[1 - \mu_n^+(u), 1 - \mu_n^-(u)]) : u \in U\}$

(4) $X \subseteq Y \Leftrightarrow$ for all $u \in U, \mu_i^-(u) \le \nu_i^-(u)$ and $\mu_i^+(u) \le \nu_i^+(u), 1 \le i \le n$.

(5) $X = Y \Leftrightarrow X \subseteq Y$ and $X \supseteq Y$.

Theorem 1. Let $X = \{u / ([\mu_1^-(u), \mu_1^+(u)], [\mu_2^-(u), \mu_2^+(u)], \cdots, [\mu_n^-(u), \mu_n^+(u)]) : u \in U\}$, $Y = \{u / ([\nu_1^-(u), \nu_1^+(u)], [\nu_2^-(u), \nu_2^+(u)], \cdots, [\nu_n^-(u), \nu_n^+(u)]) : u \in U\}$ and $Z = \{u / ([\gamma_1^-(u), \gamma_1^+(u)], [\gamma_2^-(u), \gamma_2^+(u)], \cdots, [\gamma_n^-(u), \gamma_n^+(u)]) : u \in U\}$ be three interval-valued multi-fuzzy sets of dimension *n* in *U*. Then

(1) $(X \cap Y) \cap Z = X \cap (Y \cap Z)$;

(2) $X \cap (Y \cup Z) = (X \cap Y) \cup (X \cap Z)$;

(3) $(X \cap Y) \cap Z = X \cap (Y \cap Z)$;

(4) $X \cup (Y \cap Z) = (X \cup Y) \cap (X \cup Z)$.

Proof. we only prove (1) and (2) since (3) and (4) can be proved similarly.

(1) $(X \cap Y) \cap Z$

$$= \{u / ([\mu_1^-(u) \wedge \nu_1^-(u) \wedge \gamma_1^-(u), \mu_1^+(u) \wedge \nu_1^+(u) \wedge \gamma_1^+(u)], \cdots,$$
$$[\mu_n^-(u) \wedge \nu_n^-(u) \wedge \gamma_n^-(u), \mu_n^+(u) \wedge \nu_n^+(u) \wedge \gamma_n^+(u)]) : u \in U\}$$
$$= X \cap (Y \cap Z).$$

(2) $X \cap (Y \cup Z)$

$$= \{u/([\mu_1^-(u) \wedge \nu_1^-(u) \vee \gamma_1^-(u), \mu_1^+(u) \wedge \nu_1^+(u) \vee \gamma_1^+(u)], \cdots, [\mu_n^-(u) \wedge \nu_n^-(u) \vee \gamma_n^-(u), \mu_n^+(u) \wedge \nu_n^+(u) \vee \gamma_n^+(u)]) : u \in U\}$$
$$= \{u/([(\mu_1^-(u) \wedge \nu_1^-(u)) \vee (\mu_1^-(u) \wedge \gamma_1^-(u)), (\mu_1^+(u) \wedge \nu_1^+(u)) \vee (\mu_1^+(u) \wedge \gamma_1^+(u))], \cdots, [(\mu_n^-(u) \wedge \nu_n^-(u)) \vee (\mu_n^-(u) \wedge \gamma_n^-(u)), (\mu_n^+(u) \wedge \nu_n^+(u)) \vee (\mu_n^+(u) \wedge \gamma_n^+(u))]) : u \in U\}$$
$$= (X \cap Y) \cup (X \cap Z).$$

Theorem 2. Let $X = \{u/([\mu_1^-(u), \mu_1^+(u)], [\mu_2^-(u), \mu_2^+(u)], \cdots, [\mu_n^-(u), \mu_n^+(u)]) : u \in U\}$ and $Y = \{u/([\nu_1^-(u), \nu_1^+(u)], [\nu_2^-(u), \nu_2^+(u)], \cdots, [\nu_n^-(u), \nu_n^+(u)]) : u \in U\}$ be two interval-valued multi-fuzzy sets of dimension n in U. Then

(1) $(X \cup Y)^c = X^c \cap Y^c$;

(2) $(X \cup Y) \cap X = X$;

(3) $(X \cap Y)^c = X^c \cup Y^c$;

(4) $(X \cap Y) \cup X = X$.

Proof. we only prove (1) and (2) since (3) and (4) can be proved similarly.

(1) Since $X \cup Y = \{u/([\mu_1^-(u) \vee \nu_1^-(u), \mu_1^+(u) \vee \nu_1^+(u)], \cdots, [\mu_n^-(u) \vee \nu_n^-(u), \mu_n^+(u) \vee \nu_n^+(u)]) : u \in U\}$, then

$$(X \cup Y)^c$$
$$= \{u/([1 - \mu_1^+(u) \vee \nu_1^+(u), 1 - \mu_1^-(u) \vee \nu_1^-(u)], \cdots, [1 - \mu_n^+(u) \vee \nu_n^+(u), 1 - \mu_n^-(u) \vee \nu_n^-(u)]) : u \in U\}$$
$$= \{u/([(1 - \mu_1^+(u)) \wedge (1 - \nu_1^+(u)), (1 - \mu_1^-(u)) \wedge (1 - \nu_1^-(u))], \cdots, [(1 - \mu_n^+(u)) \wedge (1 - \nu_n^+(u)), (1 - \mu_n^-(u)) \wedge (1 - \nu_n^-(u))]) : u \in U\}$$
$$= X^c \cap Y^c.$$

(2) $(\tilde{X} \cup \tilde{Y}) \cap \tilde{X} = \{u/([(\mu_1^-(u) \vee \nu_1^-(u)) \wedge \mu_1^-(u), (\mu_1^+(u) \vee \nu_1^+(u)) \wedge \mu_1^+(u)], \cdots, [(\mu_n^-(u) \vee \nu_n^-(u)) \wedge \mu_n^-(u), (\mu_n^+(u) \vee \nu_n^+(u)) \wedge \mu_n^+(u)]) : u \in U\}$

$$= \{u/([\mu_1^-(u), \mu_1^+(u)], [\mu_2^-(u), \mu_2^+(u)], \cdots, [\mu_n^-(u), \mu_n^+(u)]) : u \in U\}$$
$$= \tilde{X}$$

IV. INTERVAL-VALUED MULTI-FUZZY SOFT SET

In this section, based on the interval-valued multi-fuzzy set and soft set, we develop an extended soft set model called interval-valued multi-fuzzy soft set. we also define some operations on interval-valued multi-fuzzy soft set.

Definition 9. Let $A \subseteq E$. A pair (F, A) is called an interval-valued multi-fuzzy soft set of dimension n over U, where $F : A \to \mathrm{IVM}^n(U)$ is a mapping.

In other words, an interval-valued multi-fuzzy soft set of dimension n over U is a parameterized family of interval-valued multi-fuzzy set of the universe U.

Remark 3. (1) If A has only an element, i.e. $A = \{e\}$, then interval-valued multi-fuzzy soft set becomes interval-valued multi-fuzzy set introduced in Section 3;

(2) If $\mu_i^-(u) = \mu_i^+(u), 1 \le i \le n$ for all $e \in A$ and $u \in U$, then interval-valued multi-fuzzy soft set degenerates to multi-fuzzy soft set [2] ;

(3) If $F(e)$ is an interval-valued multi-fuzzy set of dimension 1 for each $e \in A$, then interval-valued multi-fuzzy soft set reduces to interval-valued fuzzy soft set [17];

(4) If $F(e)$ is an interval-valued multi-fuzzy set of dimension 2 with $\mu_1^+(u) + \mu_2^+(u) \le 1$ for all $e \in A$, then interval-valued multi-fuzzy soft set transforms to interval-valued intuitionistic fuzzy soft set [19].

Definition 10. Let (F, A) and (G, B) be two interval-valued multi-fuzzy soft sets of dimension *n* over *U*. Then (F, A) is called an interval-valued multi-fuzzy soft subset of (G, B), denoted by $(F, A) \subseteq (G, B)$, if

(1) $A \subseteq B$;

(2) $F(e) \subseteq G(e)$ for all $e \in A$.

Definition 11. Let (F, A) and (G, B) be two interval-valued multi-fuzzy soft sets of dimension *n* over *U*. Then (F, A) and (G, B) are said to be interval-valued multi-fuzzy soft equal if and only if $(F, A) \subseteq (G, B)$ and $(G, B) \subseteq (F, A)$.

Definition 12. The complement of an interval-valued multi-fuzzy soft set (F, A) of dimension *n* over *U* is denoted by $(F, A)^c$ and is defined by $(F, A)^c = (F^c, A)$, where $F^c : A \to \mathrm{IVM}^n(U)$ is a mapping given by $F^c(e) = (F(e))^c$.

Definition 13. Let (F, A) and (G, B) be two interval-valued multi-fuzzy soft sets of dimension *n* over *U* and $A, B \subseteq E$. We define a mapping $H : A \cup B \to \mathrm{IVM}^n(U)$ such that for all $e \in A \cup B \neq \emptyset$,

$$H(e) = \begin{cases} F(e), & \text{if } e \in A - B, \\ G(e), & \text{if } e \in B - A, \\ H(e), & \text{if } e \in A \cap B. \end{cases}$$

(1) If $H(e)=F(e)\cup G(e)$, then $(H, A\cup B)$ is called the extended union of (F,A) and (G,B), denoted by $(F,A)\ \cup(G,B)$.

(2) If $H(e)=F(e)\cap G(e)$, then $(H, A\cup B)$ is called the extended intersection of (F,A) and (G,B), denoted by $(F,A)\ \cap(G,B)$.

Theorem 3. Let $A,B,C\subseteq E$, (F,A), (G,B) and (H,C) be three interval-valued multi-fuzzy soft sets of dimension *n* over *U*. Then

(1) $(F,A)\cup(F,A)=(F,A),(F,A)\cap(F,A)=(F,A)$;

(2) $(F,A)\cup(G,B)=(G,B)\cup(F,A)$,

$(F,A)\cap(G,B)=(G,B)\cap(F,A)$;

(3) $(F,A)\cap((G,B)\cap(H,C))=(F,A)\cup((G,B)\cup(H,C)$,

(4) $((F,A)\cap(G,B))\cap(H,C)=(F,A)\cap((G,B)\cap(H,C)$;

Proof. (1) and (2) are trivial. We only prove (3) since (4) can be proved similarly.

Suppose that $(F,A)\cup((G,B)\cup(H,C))=(J,M)$ and $((F,A)\cap(G,B))\cap(H,C)=(K,N)$, so $M=N=A\cup B\cup C$. Then for all $e\in M$, it follows that $e\in A$, or $e\in B$, or $e\in C$. Without losing generality, we suppose that $e\in A$.

(i) If $e\notin B$ and $e\notin C$, then $J(e)=F(e)=K(e)$;

(ii) If $e\in B$ and $e\notin C$, then $J(e)=F(e)\cap G(e)=K(e)$;

(iii) If $e\notin B$ and $e\in C$, then $J(e)=F(e)\cap H(e)=K(e)$;

(iv) If $e\in B$ and $e\in C$, then $J(e)=(F(e)\cap G(e))\cap H(e)$ and $K(e)=F(e)\cap(G(e)\cap H(e))$. By (1) in Theorem 1, we have $(F(e)\cap G(e))\cap H(e)=F(e)\cap(G(e)\cap H(e))$, Hence, $J(e)=K(e)$.

To sum up, J and K are indeed the same set-valued mappings. Thus $(J,M)=(K,N)$, which completes the proof.

Theorem 4. Let $A,B\subseteq E$, (F,A) and (G,B) be two interval-valued multi-fuzzy soft sets of dimension *n* over *U*. Then

(1) $((F,A)\cup(G,B))^c=(F,A)^c\cap(G,B)^c$;

(2) $((F,A)\cap(G,B))^c=(F,A)^c\cup(G,B)^c$.

Proof. We only prove (1). By using a similar technique, (2) can be proved, too.

Suppose that $(F,A)\cup(G,B)=(H,C)$. Then $C=A\cup B$, For each $e\in C$ and $u\in U$, we have

$$H(e)=\begin{cases} F(e), & \text{if } e\in A-B,\\ G(e), & \text{if } e\in B-A,\\ F(e)\cup G(e), & \text{if } e\in A\cap B.\end{cases}$$

Then

$$H^c(e)=\begin{cases} F^c(e), & \text{if } e\in A-B,\\ G^c(e), & \text{if } e\in B-A,\\ (F(e)\cup G(e))^c, & \text{if } e\in A\cap B.\end{cases}$$

Again suppose that $(F,A)^c\cap(G,B)^c=(J,D)$. Then $D=A\cup B$ and for each $e\in D$, we have

$$J(e)=\begin{cases} F^c(e), & \text{if } e\in A-B,\\ G^c(e), & \text{if } e\in B-A,\\ F^c(e)\cap G^c(e), & \text{if } e\in A\cap B.\end{cases}$$

By (1) in Theorem 2, we have $F^c(e)\cap G^c(e)=(F(e)\cup G(e))^c$, i.e., $J(e)=H^c(e)$ for all $e\in A$.

Therefore, (H^c,C) and (J,M) are the same interval-valued multi-fuzzy soft sets. It follows that $((F,A)\cap(G,B))^c=(F,A)^c\cup(G,B)^c$

V. An Application in Decision Making

The rich potential of soft set theory for applications in decision making, measurement theory, game theory, etc. were shown by Molodtsov [1]. Various decision making methods to fuzzy soft set have been proposed. Roy et al. [21] gave an algorithm based on the comparison table of the decision making objects. Kong et al. [22] presented a new method based on choice values of different objects. Later, Feng et al. [23] developed a novel method by using level soft sets, which can solve the decision making problems that cann't be dealt with using the methods in [14,22]. Furthermore, Feng's method has been successfully applied to many extended soft set models based on decision making [14,24,25]. In this section, we will present a decision making method to interval-valued multi-fuzzy soft set based on Feng's algorithm [23] and Yang's methods [15].

Let $U=\{u_1,u_2,\cdots,u_m\}$ and (F,A) be a normalized interval-valued multi-fuzzy soft set of dimension *n* over *U*. For each $e\in A$,

$$\begin{aligned}F(e)=\{&u_1/([\mu_1^-(u_1),\mu_1^+(u_1)],\cdots,[\mu_n^-(u_1),\mu_n^+(u_1)]),\\ &u_2/([\mu_1^-(u_2),\mu_1^+(u_2)],\cdots,[\mu_n^-(u_2),\mu_n^+(u_2)],\cdots,\\ &u_m/([\mu_1^-(u_m),\mu_1^+(u_m)]),\cdots,[\mu_n^-(u_m),\mu_n^+(u_m)])\},\end{aligned}$$

which can be expressed by the following matrix

$$F(e)=\begin{pmatrix}[\mu_1^-(u_1),\mu_1^+(u_1)] & \cdots & [\mu_1^-(u_1),\mu_1^+(u_1)]\\ [\mu_1^-(u_2),\mu_2^+(u_1)] & \cdots & [\mu_2^-(u_2),\mu_2^+(u_2)]\\ \vdots & \vdots & \vdots\\ [\mu_1^-(u_m),\mu_1^+(u_m)] & \cdots & [\mu_n^-(u_m),\mu_n^+(u_m)]\end{pmatrix}$$

Suppose that $w(e)=(w_1,w_2,\cdots,w_n)^T$ is the relative weight vector of parameter e and $\sum_{i=1}^{n} w_i=1$, we define an induced interval-valued fuzzy set $\mu_{F(e)}$ with respect to e in U as follows

$$\mu_{F(e)}=\begin{pmatrix}[\mu_1^-(u_1),\mu_1^+(u_1)] & \cdots & [\mu_1^-(u_1),\mu_1^+(u_1)]\\ [\mu_1^-(u_2),\mu_2^+(u_1)] & \cdots & [\mu_2^-(u_2),\mu_2^+(u_2)]\\ \vdots & \vdots & \vdots\\ [\mu_1^-(u_m),\mu_1^+(u_m)] & \cdots & [\mu_n^-(u_m),\mu_n^+(u_m)]\end{pmatrix}\begin{pmatrix}w_1\\ w_2\\ \vdots\\ w_n\end{pmatrix}$$

$$=\begin{pmatrix}\left[\sum_{i=1}^{n} w_i\mu_i^-(u_1),\sum_{i=1}^{n} w_i\mu_i^+(u_1)\right]\\ \left[\sum_{i=1}^{n} w_i\mu_i^-(u_2),\sum_{i=1}^{n} w_i\mu_i^+(u_2)\right]\\ \vdots\\ \left[\sum_{i=1}^{n} w_i\mu_i^-(u_m),\sum_{i=1}^{n} w_i\mu_i^+(u_m)\right]\end{pmatrix} \quad (1)$$

That is to say, if the weight vector $w(e)$ is given, then we can obtain an induced interval-valued fuzzy set $\mu_{F(e)}$ from an interval-valued multi-fuzzy set $F(e)$. Thus, by using this method, we can change an interval-valued multi-fuzzy soft set into an induced interval-valued fuzzy soft set. In the following, we introduce the concept of level soft set of interval-valued fuzzy soft set.

Definition 14. Let $\mathsf{F}=(F_w,A)$ be an interval-valued fuzzy soft set over U, where $A\in E$ and E is the parameter set. For $\alpha\in Int([0,1])$, the α-level soft set of F is a crisp soft set $L(\mathsf{F},\alpha)=(F_w^{\alpha},A)$ defined by

$$F_w^{\alpha}(e)=L(\mathsf{F}(e),\alpha)=\{u\in U:\mu_{F_w^{\alpha}(e)}(u)\ge\alpha\}$$
$$=\{u\in U:\mu^-_{F_w^{\alpha}(e)}(u)\ge\alpha^-,\mu^+_{F_w^{\alpha}(e)}(u)\ge\alpha^+\}$$

In Definition 14 the threshold value assigned to each parameter is always the constant value $\alpha\in Int([0,1])$. However, in many practical problems, decision makers need to impose different thresholds on different parameters. To overcome the difficulty, we replace a constant value by a function as the thresholds on membership values.

Definition 15. Let $\mathsf{F}=(F_w,A)$ be an interval-valued fuzzy soft set over U, where $A\in E$ and E is the parameter set. Let $\lambda: A\to Int([0,1])$ be an interval-valued fuzzy set in A, which is called a threshold interval-valued fuzzy set. The level soft set of F with respect to λ is a crisp soft set $L(\mathsf{F},\lambda)=(F_w^{\lambda},A)$ defined by

$$F_w^{\lambda}(e)=L(\mathsf{F}(e),\lambda(e))=\{u\in U:\mu_{F_w^{\lambda}(e)}(u)\ge\lambda(e)\}$$
$$=\{u\in U:\mu^-_{F_w^{\lambda}(e)}(u)\ge\lambda^-(e),\mu^+_{F_w^{\lambda}(e)}(u)\ge\lambda^+(e)\}$$

Note that the threshold interval-valued fuzzy set can be given arbitrarily by the decision maker according to his/her preference and interest in decision making process. For example, to an ordinary decision maker, let $\mathsf{F}=(F_w,A)$ be an interval-valued fuzzy soft set over U, we can define an interval-valued fuzzy set $mid_{\mathsf{F}}: A\to Int([0,1])$ by

$$\mu^-_{mid_{\mathsf{F}}}(e)=\frac{1}{|U|}\sum_{u\in U}\mu^-_{F_w(e)}(u),$$
$$\mu^+_{mid_{\mathsf{F}}}(e)=\frac{1}{|U|}\sum_{u\in U}\mu^+_{F_w(e)}(u) \quad (2)$$

for all $e\in A$, where $|\mathrm{U}|$ represents the cardinality of U.

The interval-valued fuzzy set mid_{F} is called the mid-threshold of the interval-valued fuzzy soft set F. In addition, the level soft set of F with respect to the mid-threshold interval-valued fuzzy set mid_{F}, denoted by $L(\mathsf{F},mid_{\mathsf{F}})$, is called the mid-level soft set of F. In the following discussions, the mid-level decision rule will mean using the mid-threshold and considering the mid-level soft set.

Algorithm I:

Step 1. Input the interval-valued multi-fuzzy soft set (F,A) and the relative weight vector $w(e_i)$ of parameter e_i for all $e_i\in A$.

Step 2. Change the (F,A) into the normalized interval-valued multi-fuzzy soft set.

Step 3. Calculate the induced interval-valued fuzzy soft set $\mathsf{F}=(F_w,A)$ by Eq. (1).

Step 4. Input a threshold interval-valued fuzzy set $\lambda: A\to Int([0,1])$ (or give a threshold value $\alpha\in Int([0,1])$; or choose the mid-level decision rule) for decision making.

Step 5. Calculate the level soft set $L(\mathsf{F},\lambda)$ of F with respect to the threshold interval-valued fuzzy set λ (or the $\alpha-$ level soft set $L(\mathsf{F},\alpha)$; or the mid-level soft set $L(\mathsf{F},mid_{\mathsf{F}})$).

Step 6. Present the level soft set $L(\mathsf{F},\lambda)$ (or $L(\mathsf{F},\alpha)$; or $L(\mathsf{F},mid_{\mathsf{F}})$) in tabular form. For each $u_i\in U$, compute the choice value $c(i)$ of $u(i)$.

Step 7. The optimal decision is to select $u(k)$ if $c_k=max\{c_i:u_i\in U\}$.

Step 8. If k has more than one value then any one of $u(k)$ may be chosen.

Remark 4. It should be pointed out that the threshold can be adjusted by the user (decision maker) interactively to fulfill the real needs better. Especially when there are too many “optimal choices" to be chosen, we may go back to the fourth step and change the threshold (or decision rule) such that only one optimal choice remains in the end.

Remark 5. The goal of designing Algorithm I is to solve decision making problems based on interval-valued multi-fuzzy soft set by using level soft set. By using Algorithm I, we actually do not deal directly with interval-valued multi-fuzzy soft set, but only need to cope with the induced interval-valued fuzzy soft set and finally the crisp level soft set in decision making process. It makes the proposed algorithm simpler in computational complexity and thus easier for application in real problems.

To illustrate the basic idea of Algorithm 1, let us consider the following example (adapted from [15]).

Example I. Assume that $U=\{u_1,u_2,u_3,u_4,u_5\}$ is the universe consisting of five kinds of colored drawing paper for engineering. Suppose the parameter set $A=\{e_1,e_2,e_3\}$, i.e., we have three criteria to evaluate the performance of these papers, where e_1 stands for ‘thickness’ which includes three levels: thick, average and thin, e_2 stands for ‘color’ which consists of red, green and blue, and e_3 stands for ‘ingredient’ which is made from cellulose, hemicellulose and lignin. Suppose that

$$
\begin{aligned}
F(e_1)=\{&u_1/([0.12,0.24], [0.21,0.45], [0.15,0.28]),\\
&u_2/([0.25,0.38], [0.18,0.35], [0.08,0.24]),\\
&u_3/([0.23,0.42], [0.10,0.22], [0.15,0.34]),\\
&u_4/([0.05,0.21], [0.16,0.32], [0.26,0.43]),\\
&u_5/([0.28,0.34], [0.07,0.13], [0.32,0.52])\};\\
F(e_2)=\{&u_1/([0.16,0.28], [0.30,0.38], [0.25,0.34]),\\
&u_2/([0.20,0.35], [0.28,0.42], [0.12,0.23]),\\
&u_3/([0.15,0.25], [0.23,0.29], [0.32,0.46]),\\
&u_4/([0.24,0.36], [0.13,0.28], [0.16,0.33]),\\
&u_5/([0.28,0.42], [0.21,0.30], [0.16,0.28])\};\\
F(e_3)=\{&u_1/([0.23,0.34], [0.17,0.25], [0.21,0.38]),\\
&u_2/([0.32,0.45], [0.13,0.20], [0.26,0.34]),\\
&u_3/([0.20,0.25], [0.15,0.21], [0.24,0.32]),\\
&u_4/([0.16,0.19], [0.27,0.32], [0.35,0.46]),\\
&u_5/([0.10,0.18], [0.25,0.32], [0.36,0.47])\}.
\end{aligned}
$$

Suppose that a company would like to select one of them depending on its performance, and the company has imposed the following weights for the parameters in A: for the parameter ‘thickness’, $w(e_1)=(0.3, 0.1, 0.6)$, for the parameter ‘color’, $w(e_2)=(0.5, 0.3, 0.2)$, and for the parameter ‘ingredient’, $w(e_3)=(0.3, 0.3, 0.4)$.

Obviously, (F, A) is normalized. Thus, by Eq. (1), we can get an induced interval-valued fuzzy soft set $\mathsf{F}=(F_w, A)$, whose tabular representation is given in TABLE I.

TABLE I

THE INDUCED INTERVAL-VALUED FUZZY SOFT SET F

U	e_1 $w(e_1)=$ $(0{:}3;\,0{:}1;\,0{:}6)$	e_2 $w(e_2)=$ $(0{:}5;\,0{:}3;\,0{:}2)$	e_3 $w(e_3)=$ $(0{:}3;\,0{:}3;\,0{:}4)$
u_1	[0.147,0.285]	[0.220,0.322]	[0.200,0.316]
u_2	[0.141,0.293]	[0.208,0.347]	[0.226,0.317]
u_3	[0.169,0.352]	[0.208,0.304]	[0.192,0.255]
u_4	[0.187,0.353]	[0.191,0.330]	[0.261,0.323]
u_5	[0.283,0.427]	[0.235,0.356]	[0.238,0.323]

As an adjustable approach, one can use different rules (or the thresholds) in decision making problem. For example, if we deal with this problem by mid-level decision rule, then the mid-threshold mid_{F} of F is an interval-valued fuzzy set in A. By Eq. (1), we have

$$
\begin{aligned}
\mu^-_{mid_{\mathsf{F}}}(e_1)&=(0.147+0.141+0.169+0.187+0.283)/5=0.1854,\\
\mu^+_{mid_{\mathsf{F}}}(e_1)&=(0.285+0.293+0.352+0.353+0.427)/5=0.3420,\\
\mu^-_{mid_{\mathsf{F}}}(e_2)&=(0.220+0.208+0.208+0.191+0.235)/5=0.2124,\\
\mu^+_{mid_{\mathsf{F}}}(e_2)&=(0.322+0.347+0.304+0.330+0.356)/5=0.3318,\\
\mu^-_{mid_{\mathsf{F}}}(e_3)&=(0.200+0.226+0.192+0.261+0.238)/5=0.2234,\\
\mu^+_{mid_{\mathsf{F}}}(e_3)&=(0.316+0.317+0.255+0.323+0.323)/5=0.3068.
\end{aligned}
$$

Therefore, the mid-threshold mid_{F} can be expressed as

$$
\begin{aligned}
mid_{\mathsf{F}}=\{&e_1/[0.1854, 0.3420],\\
&e_2/[0.2124, 0.3318],\\
&e_3/[0.2234, 0.3068]\}
\end{aligned}
$$

By Definition 15, the mid-level soft set of F is a crisp soft set $L(\mathsf{F}, mid_{\mathsf{F}}))=(F_{mid_{\mathsf{F}}}, A)$ and it can be calculated as follows:

$$
\begin{aligned}
F_{mid_{\mathsf{F}}}(e_1)&=L(F_w(e_1),[0.1854, 0.342])=\{u_4,u_5\},\\
F_{mid_{\mathsf{F}}}(e_2)&=L(F_w(e_2),[0.2124, 0.3318])=\{u_5\},\\
F_{mid_{\mathsf{F}}}(e_3)&=L(F_w(e_3),[0.2234, 0.3234])=\{u_2,u_4,u_5\}.
\end{aligned}
$$

TABLE II gives the tabular representation of the mid-level soft set $L(\mathsf{F}, mid_{\mathsf{F}}))$ with choice values. If $u_i\in F_{mid_{\mathsf{F}}}(e_j)$, then $u_{ij}=1$, otherwise $u_{ij}=0$, where u_{ij} are the entries in TABLE II.

TABLE II

THE MID-LEVEL SOFT SET $L(\mathrm{F}, mid_{\mathrm{F}}))$ WITH CHOICE

U	e_1	e_2	e_3	choice value
$u1$	0	0	0	0
$u2$	0	0	1	1
$u3$	0	0	0	0
$u4$	1	0	1	2
$u5$	1	1	1	3

In TABLE II, we can calculate choice values as follows:

$$c_5 = \sum_{j=1}^{3} F_{mid_{\mathrm{F}}}(e_j)(u_i) = 1+1+1 = 3.$$

Similarly, we have $c_4 = 2$, $c_3 = 0$, $c_2 = 1$ and $c_1 = 0$.

From TABLE II, it follows that the maximum choice value is 3 and so the optimal decision is to select u_5. Therefore, the company should select u_5 as the best colored drawing paper for engineering after specifying weights for different parameters.

VI. CONCLUSION

In this work, we have extended multi-fuzzy set to interval-valued multi-fuzzy set and developed a new hybrid model called interval-valued multi-fuzzy soft set by combining interval-valued multi-fuzzy set and soft set models. It has been pointed out that the interval-valued multi-fuzzy soft set is an extension of many existing soft set models, such as interval-valued multi-fuzzy set, multi-fuzzy soft set, interval-valued fuzzy soft set and interval-valued intuitionistic fuzzy soft set. We also have defined some basic operations on interval-valued multi-fuzzy soft set and discussed some basic properties of those operations. Finally, a decision making approach based on interval-valued multi-fuzzy soft set have been proposed, and a practical example have been provided to illustrate the developed approach. We hope that our work would be useful to handle some other realistic uncertain problems.

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